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Finite Quantum Field Theory in Noncommutative Geometry

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We describe a self-interacting scalar field on a truncated sphere and perform the quantization using the functional (path) integral approach. The theory possesses full symmetry with respect to the isometries of the sphere. We explicitly show that the model is finite and that UV regularization automatically takes place.

1. INTRODUCTION

The basic ideas of noncommutative geometry were developed in Connes (1986, 1990) and in the form of matrix geometry in Dubois-Violette (1988) and Dubois-Violette *et al.* (1990). The applications to physical models were presented in Connes (1990) and Coquereaux *et al.* (1991), where the noncommutativity was in some sense minimal: the Minkowski space was not extended by some standard Kaluza–Klein manifold describing internal degrees of freedom, but by just two noncommutative points. This led to new insight into the $SU(2)_L \otimes U(1)_R$ symmetry of the standard model of electroweak interactions. The model was further extended in Chamseddine *et al.* (1992), extending the Minkowski space by a pseudo-Riemannian manifold, and thus including gravity. Such models, of course, do not lead to UV regularization, since they do not introduce any space-time short-distance behavior.

To achieve UV regularization one should introduce noncommutativity into the genuine space-time manifold in the relativistic case, or into the space manifold in the Euclidean version. One of the simplest locally Euclidean manifolds is the sphere S^2. Its noncommutative (fuzzy) analog was described by Madore (1991, 1992, n.d.) in the framework of matrix geometry. A more

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general construction of some noncommutative homogeneous spaces was described in Gross and Prešnajder (1993) using the coherent states technique.

The first attempt to construct fields on a truncated sphere was presented in Madore (1992, n.d.) and Grosse and Madore (1992) within the matrix formulation. Using a more general approach, Grosse *et al.* (n.d.-a,b) investigated in detail the classical spinor field on truncated S^2.

In this article we investigate the quantum scalar field Φ on truncated S^2. We explicitly demonstrate that UV regularization automatically appears within the context of noncommutative geometry. We introduce only those notions of noncommutative geometry that we need in our approach. In Section 2 we define the noncommutative sphere and derivation and integration on it. In Section 3 we introduce the scalar self-interacting field Φ on the truncated sphere and the field action. Further, using Feynman (path) integrals, we perform the quantization of the model in question. Finally, Section 4 contains a brief discussion and concluding remarks.

2. NONCOMMUTATIVE TRUNCATED SPHERE

2.1. The infinite-dimensional algebra $\mathcal{A}_\infty$ of polynomials generated by $x. = (x_1, x_2, x_3) \in \mathbf{R}^3$, with the defining relations

$$[x_i, x_j] = 0, \qquad \sum_{i=1}^{3} x_i^2 = \rho^2 \tag{1}$$

contains all the information about the standard unit sphere S^2 embedded in $\mathbf{R}^3$. In terms of the spherical angles θ and φ, we have

$$x_\pm = x_1 \pm ix_2 = \rho e^{\pm i\varphi} \sin\theta, \qquad x_3 = \rho\cos\theta \tag{2}$$

As a noncommutative analog of $\mathcal{A}_\infty$ we take the algebra $\mathcal{A}_N$ generated by $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$, with the defining relations

$$[\hat{x}_i, \hat{x}_j] = i\lambda\epsilon_{ijk}\hat{x}_k, \qquad \sum_{i=1}^{3} \hat{x}_i^2 = \rho^2 \tag{3}$$

The real parameter $\lambda > 0$ characterizes the noncommutativity (later it will be related to N). In terms of $\hat{X}_i = (1/\lambda)\hat{x}_i$, $i = 1, 2, 3$, equations (3) are changed to

$$[\hat{X}_i, \hat{X}_j] = i\epsilon_{ijk}\hat{X}_k, \qquad \sum_{i=1}^{3} \hat{X}_i^2 = \rho^2\lambda^{-2} \tag{4}$$

or putting $X_\pm = X_1 \pm iX_2$, we obtain

$$[\hat{X}_3, \hat{X}_\pm] = \hat{X}_\pm, \qquad [\hat{X}_+, \hat{X}_-] = 2\hat{X}_3 \tag{5}$$

and

$$C = \hat{X}_3^2 + \frac{1}{2}(\hat{X}_+\hat{X}_- + \hat{X}_-\hat{X}_+) = \rho^2\lambda^{-2} \tag{6}$$

We shall realize equations (4), or equivalently equations (5) and (6), as relations in some suitable irreducible unitary representations of the $SU(2)$ group. It is useful to perform this construction using the Wigner–Jordan realization of the generators $\hat{X}_i$, $i = 1, 2, 3$, in terms of two pairs of annihilation and creation operators A_α, A_α^*, $\alpha = 1, 2$, satisfying

$$[A_\alpha, A_\beta] = [A_\alpha^*, A_\beta^*] = 0, \qquad [A_\alpha, A_\beta^*] = \delta_{\alpha,\beta} \tag{7}$$

and acting in the Fock space $\mathcal{F}$ spanned by the normalized vectors

$$|n_1, n_2\rangle = \frac{1}{(n_1!n_2!)^{1/2}}(A_1^*)^{n_1}(A_2^*)^{n_2}|0\rangle \tag{8}$$

where $|0\rangle$ is the vacuum defined by $A_1|0\rangle = A_2|0\rangle = 0$. The operators $\hat{X}_\pm$ and $\hat{X}_3$ take the form

$$\hat{X}_+ = 2A_1^*A_2, \qquad \hat{X}_- = 2A_2^*A_1, \qquad \hat{X}_3 = \frac{1}{2}(N_1 - N_2) \tag{9}$$

where $N_\alpha = A_\alpha^*A_\alpha$, $\alpha = 1, 2$. Restricting ourselves to the $(N + 1)$-dimensional subspace

$$\mathcal{F}_N = \{|n_1, n_2\rangle \in \mathcal{F}\} \tag{10}$$

we obtain for any given $N = 0, 1, 2, \ldots$ The irreducible unitary representation in which the Casimir operator (6) has the value

$$C = \frac{N}{2}\left(\frac{N}{2} + 1\right) \tag{11}$$

i.e., λ and N are related as

$$\rho\lambda^{-1} = \left[\frac{N}{2}\left(\frac{N}{2} + 1\right)\right]^{1/2} \tag{12}$$

The states $|n_1, n_2\rangle$ are eigenstates of the operator X_3, whereas X_+ and X_- are raising and lowering operators, respectively:

$$X_3|n_1, n_2\rangle = \frac{n_1 - n_2}{2}|n_1, n_2\rangle$$

$$X_+|n_1, n_2\rangle = 2[(n_1 + 1)n_2]^{1/2}|n_1 + 1, n_2 - 1\rangle$$

$$X_-|n_1, n_2\rangle = 2[n_1(n_2 + 1)]^{1/2}|n_1 - 1, n_2 + 1\rangle \tag{13}$$

Since X_i: $\mathcal{F}_N \to \mathcal{F}_N$, we have

$$\dim \mathcal{A}_N \leq (N+1)^2 \tag{14}$$

2.2. As a next step we extend the notions of integration and derivation to the truncated case. The standard integral on S^2

$$I_\infty(F) = \frac{1}{4\pi}\int d\Omega\, F(x) = \frac{1}{4\pi}\int_{-\pi}^{+\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta\, F(\theta, \varphi) \tag{15}$$

is uniquely defined if it is fixed for the monomials $F(x) = x_+^l x_-^m x_3^n$. It is obvious that $I_\infty(x_+^l x_-^m x_3^n) = 0$ for $l \neq m$, and that $x_+^l x_-^l x_3^n = \rho^{2l+n}\sin^{2l}\theta\cos^n\theta$ is a polynomial in $\cos\theta = x_3$. An easy calculation gives

$$I_\infty(x_3^{2n+1}) = 0, \qquad I_\infty(x_3^{2n}) = \frac{\rho^{2n}}{2n+1}$$

for $n = 0, 1, 2, \ldots$. Putting $\xi = \rho^{-1}x_3 = \cos\theta$, we see that

$$I_\infty(\xi^n) = \frac{1}{2}\int_{-1}^{+1} d\xi\, \xi^n \tag{16}$$

These relations algebraically define the integration in $\mathcal{A}_\infty$.

In the noncommutative case we put

$$I_N(F) = \frac{1}{N+1}\,\mathrm{Tr}[F(\hat{x})] \tag{17}$$

for any polynomial $F(\hat{x}) \in \mathcal{A}_N$ in $\hat{x}_i$, $i = 1, 2, 3$, where the trace is taken in $\mathcal{F}_N$. Again, the integrals $I(\hat{x}_+^l \hat{x}_-^m \hat{x}_3^n) = 0$ for $l \neq m$, since

$$\hat{x}_+^l \hat{x}_-^m \hat{x}_3^n |n_1, n_2\rangle \sim |n_1 + l - m, n_2 + m - l\rangle$$

Much as before, $\hat{x}_+^l \hat{x}_-^l \hat{x}_3^n$ can be expressed using equations (5) and (6) as a polynomial in $\hat{x}_3$. The equation

$$\hat{x}_3^n |n_1, n_2\rangle = \left(\lambda\, \frac{n_1 - n_2}{2}\right)^n |n_1, n_2\rangle \tag{18}$$

gives

$$I_N(\hat{x}_3^n) = \sum_{k=0}^{N} \frac{\rho^n}{N+1}\, \xi_k^n \tag{19}$$

where $\xi_k = [N/(N+2)]^{1/2}(2k/N - 1)$. The formula (19) can be rewritten as a Stieltjes integral with the stair-shape measure $\mu(\xi)$ in the interval $(-1, +1)$ with steps at the points ξ_k:

$$I_N(\xi^n) = \int_{-1}^{+1} d\mu(\xi)\, \xi^n = \sum_{k=0}^{N} \frac{1}{N+1} \xi_k^n \tag{20}$$

Obviously, $I_N(\hat{x}_3^{2n+1}) = 0$, and

$$I_N(\hat{x}_3^{2n}) = \frac{\rho^{2n}}{(N/2)^n(N/2+1)^n(N+1)} \sum_{k=0}^{N} \left(\frac{2k-N}{2}\right)^{2n}$$

Using the formula [see, e.g., Grosse *et al.* (n.d.-b), p. 597, equation (16)]

$$\sum_{k=0}^{N} (k+a)^m = \frac{1}{m+1} [B_{m+1}(N+1+a) - B_{m+1}(a)]$$

where $B_m(x)$ are Bernoulli polynomials, we obtain

$$I_N(\hat{x}_3^{2n}) = \frac{\rho^{2n}}{2n+1} C(N, n) \tag{21}$$

Here,

$$C(N, n) = \frac{B_{2n+1}(N/2+1) - B_{2n+1}(-N/2)}{(N/2)^n(N/2+1)^n(N+1)} \tag{22}$$

represents a noncommutative correction. Since the Bernoulli polynomials are normalized as

$$B_m(x) = x^m + \text{lower powers}$$

we see that

$$C(N, n) = 1 + o(1/N) \tag{23}$$

i.e., in the limit $N \to \infty$ we recover the commutative result.

The scalar product in $\mathcal{A}_\infty$ can be introduced as

$$(F_1, F_2)_\infty = I_\infty(F_1^* F_2) \tag{24}$$

and similarly in $\mathcal{A}_N$ we put

$$(F_1, F_2)_N = I_N(F_1^* F_2) \tag{25}$$

2.3. The vector fields describing motions on S^2 are linear combinations (with the coefficients from $\mathcal{A}_\infty$) of the differential operators acting on any $F \in \mathcal{A}_\infty$ as follows:

$$J_i F = \frac{1}{i} \epsilon_{ijk} x_j \frac{\partial F}{\partial x_k} \tag{26}$$

In particular,

$$J_i x_j = i\epsilon_{ijk} x_k \tag{27}$$

The operators J_i, $i = 1, 2, 3$, satisfy in $\mathcal{A}_\infty$ the $su(2)$ algebra commutation relations

$$[J_i, J_j] = i\epsilon_{ijk} J_k \tag{28}$$

or for $J_\pm = J_1 \pm iJ_2$ they take the form

$$[J_3, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = 2J_3 \tag{29}$$

The operators J_i are self-adjoint with respect to the scalar product (24).

In the noncommutative case the operators J_i act on any element F from the algebra $\mathcal{A}_N$ in the following way:

$$J_i F = [X_i, F] \tag{30}$$

In particular,

$$J_i \hat{x}_j = i\epsilon_{ijk} \hat{x}_k \tag{31}$$

The operators J_i satisfy $su(2)$ algebra commutation relations and are self-adjoint with respect to the scalar product (25).

The functions

$$\Psi_{ll}(\hat{x}) = c_l \hat{x}_+^l \tag{32}$$

are the highest weight vectors in $\mathcal{A}_N$ for $l = 0, 1, \ldots, N$, since

$$J_+ \Psi_{ll}(\hat{x}) = \lambda^l [\hat{X}_+, \hat{X}_+^l] = 0 \tag{33}$$

For all $l > N$ one has $\hat{x}_+^l = 0$ in $\mathcal{A}_N$. The normalization factor c_l is fixed by the condition

$$1 = \|\Psi_{ll}\|^2 = (\Psi_{ll}, \Psi_{ll})_N = |c_l|^2 I_N(\hat{x}_-^l \hat{x}_+^l)$$

and is given by the formula [Prudnikov *et al.* (1981), p. 618, equation (36)]

$$\rho^{2l} c_l^2 = \frac{(2l + 1)!!}{(2l)!!} \, \frac{(N + 1)N^l (N + 2)^l (N - l)!}{(N + l + 1)!} \tag{34}$$

The second factor on the right-hand side represents a noncommutative correction. For $N \to \infty$ it approaches 1. The other normalized functions Ψ_{lm}, $m = 0, \pm 1, \ldots, \pm l$, in the irreducible representation containing Ψ_{ll} are given as

$$\Psi_{lm} = \left[\frac{(l + m)!}{(l - m)!(2l)!}\right]^{1/2} J_-^{l-m} \Psi_{ll} \tag{35}$$

The normalization factor on the right-hand side is the standard one, independent of N. The functions Ψ_{lm} are eigenfunctions of the operators J_i^2 and J_3:

$$J_i^2 \Psi_{lm} = l(l+1)\Psi_{lm}$$
$$J_3 \Psi_{lm} = m\Psi_{lm} \tag{36}$$

We see that $\mathcal{A}_N$ contains all $SU(2)$ irreducible representations with the 'orbital momentum' $l = 0, 1, \ldots, N$, the lth representation has the dimension $2l + 1$, and consequently

$$\dim \mathcal{A}_N \geq \sum_{n=0}^{N} (2l + 1) = (N + 1)^2 \tag{37}$$

Comparing this with equation (14), we see that $\mathcal{A}_N$ contains no other representations, i.e.,

$$\mathcal{A}_N = \bigoplus_{l=0}^{N} \mathcal{A}_{(l)} \tag{38}$$

where $\mathcal{A}_{(l)}$ denotes the representation space of the lth representation spanned by the functions Ψ_{lm}, $m = 0, \pm 1, \ldots, \pm l$. In particular, $\dim \mathcal{A}_N = (N + 1)^2$.

3. SCALAR FIELD ON THE TRUNCATED SPHERE

3.1. The Euclidean field action for a real self-interacting scalar field Φ on a standard sphere S^2 is given as

$$S[\Phi] = \frac{1}{4\pi} \int_{S^2} d\Omega \; [(J_i\Phi)^2 + \mu^2(\Phi)^2 + V(\Phi)]$$
$$= I_\infty(\Phi J_i^2 \Phi + \mu^2(\Phi)^2 + V(\Phi)) \tag{39}$$

where

$$V(\Phi) = \sum_{k=0}^{2K} g_k \Phi^k \tag{40}$$

is a polynomial with $g_{2K} \geq 0$ (and we have explicitly indicated the mass term).

The quantum mean value of some polynomial field functional $F[\Phi]$ is defined as the functional integral

$$\langle F[\Phi] \rangle = \frac{\int D\Phi e^{-S[\Phi]} F[\Phi]}{\int D\Phi e^{-S[\Phi]}} \tag{41}$$

where $D\Phi = \Pi_x \, d\Phi(x)$. Alternatively, we can expand the field into spherical functions

$$\Phi(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} a_{lm} Y_{lm}(x) \tag{42}$$

satisfying

$$J_i^2 Y_{lm} = l(l + 1)Y_{lm}$$

Here the complex coefficients a_{lm} obey

$$a_{l,-m} = (-1)^m a_{lm}^* \tag{43}$$

which guarantees the reality condition $\Phi^*(\hat{x}) = \Phi(\hat{x})$. We can put $D\Phi = \Pi_l da_{l0} \Pi_{lm} da_{lm} da_{lm}^*$, $l = 0, 1, \ldots, m = 1, \ldots, l$. Both expressions for $D\Phi$ are only formal. The measure in the functional integral can mathematically be rigorously defined (see, e.g., Simon, 1974), but we shall not follow this direction.

Such problems do not appear in the noncommutative case, where the scalar field $\Phi(\hat{x})$ is an element of the algebra $\mathcal{A}_N$, and consequently can be expanded as

$$\Phi(\hat{x}) = \sum_{l=0}^{N} \sum_{m=-l}^{+l} a_{lm}\Psi_{lm}(\hat{x}) \tag{44}$$

where $\Psi_{lm}(\hat{x})$ satisfy in $\mathcal{A}_N$ the equation

$$J_i^2\Psi_{lm} = l(l + 1)\Psi_{lm}$$

and are orthonormal with respect to the scalar product (25). The coefficients a_{lm} are again restricted by the condition (43).

The action in the noncommutative case is defined as (see also Madore, 1992, n.d.)

$$S[\Phi] = I_N(\Phi J_i^2\Phi + \mu^2(\Phi)^2 + V(\Phi)) \tag{45}$$

and it is a polynomial in the variables a_{lm}, $l = 0, 1, \ldots, N$, $m = 0, \pm 1, \ldots, \pm l$. The measure $D\Phi = \Pi_l da_{l0} \Pi_{lm} da_{lm} da_{lm}^*$, $l = 0, 1, \ldots, N$, $m = 1, \ldots, l$, in the quantum mean value (41) is the usual Lebesgue measure, since the product is now finite. It is equivalent to one described in Madore (1992, n.d.). The quantum mean values are well defined for any analytic functional $F[\Phi]$.

Under rotations

$$\hat{x}_i \to \hat{x}_i' = \sum_j R_{ij}(\alpha, \beta, \gamma)\hat{x}_j \tag{46}$$

specified by the Euler angles α, β, γ, the field transforms as

$$\Phi(\hat{x}) \to \Phi(\hat{x}') = \sum_{l=0}^{N} \sum_{m=-l}^{+l} a_{lm}\Psi_{lm}(\hat{x}') \tag{47}$$

Using the transformation rule for the functions Ψ_{lm} (see, e.g., Vilenkin, 1965)

$$\Psi_{lm'}(\hat{x}') = \sum_{m'} D^l_{m'm}(\alpha, \beta, \gamma)\Psi_{lm}(\hat{x}) \tag{48}$$

we obtain the transformation rule for the coefficients a_{lm},

$$a_{lm} \rightarrow a'_{lm'} = \sum_{m} D^l_{m'm}(\alpha, \beta, \gamma)a_{lm} \tag{49}$$

The last equation is an orthogonal transformation not changing the measure $D\Phi$ (see, e.g., Vilenkin, 1965).

We define the Schwinger functions as follows:

$$S_n(F) = \langle F_n[\Phi]\rangle \tag{50}$$

where

$$\begin{aligned} F_n[\Phi] &= \sum \alpha_{l_1m_1\cdots l_nm_n} a_{l_1m_1} \cdots a_{l_nm_n} \\ &\equiv \sum \alpha_{l_1m_1\cdots l_nm_n}(\Psi_{l_1m_1}, \Phi)_N \cdots (\Psi_{l_nm_n}, \Phi)_N \end{aligned} \tag{51}$$

The functions (49) satisfy the following Osterwalder–Schrader axioms:

(OS1) *Hermiticity:*

$$S_n^*(F) = S_n(\Theta F) \tag{52}$$

where ΘF is the involution

$$\Theta F_n[\Phi] = \sum \alpha^*_{l_1-m_1\cdots l_n-m_n}(-1)^{m_1+\cdots m_n} a_{l_1m_1}\cdots a_{l_nm_n}$$

(OS2) *Covariance:*

$$S_n(F) = S_n(\mathcal{R}F) \tag{53}$$

where $\mathcal{R}F$ is a mapping induced by equation (49).

(OS3) *Reflection positivity:*

$$\sum_{n,m\in\mathcal{I}} S_{n+m}(\Theta F_n \otimes F_m) \geq 0 \tag{54}$$

(OS4) *Symmetry:*

$$S_n(F) = S_n(\pi F) \tag{55}$$

where πF is a functional obtained from F by arbitrary permutation of the a_{lm} in equation (51).

Note. The positivity axiom (54) can be rewritten as $\langle F^*F\rangle \geq 0$, $F = \Sigma_{n\in\mathcal{I}} F_n$. In fact, the standard formulation of the (OS3) axiom requires the specification of the support of the functionals F_n. In our case the axiom holds in the 'strong' sense, i.e., without the specification. We expect, however, that in the continuum limit ($N \to \infty$) the issue will emerge. We do not include the last Osterwalder–Schrader axiom, the cluster property, since the compact manifold requires a special treatment (however, it can be recovered in the limit where the radius of the sphere grows to infinity, but this is beyond the scope of this paper).

3.2. In many practical applications the perturbative results are sufficient. Interpreting the term $V(\Phi)$ as a perturbation, we present below as an illustration the Feynman rules for the model in question. We give the Feynman rules in the (lm) representation defined by the expansions (42) and (43). The diagrams are constructed from the following:

(i) *External vertices* assigned to any operator a_{lm} appearing in the functional $F[\Phi]$.

(ii) *Internal vertices* given by the expansion of $V(\Phi)$ in terms of $a_{l_1 m_1} \cdots a_{l_k m_k}$.

This gives the following Feynman rules:

(a) *Propagator:*

$$2\langle a_{lm} a^*_{l'm'} \rangle = \frac{1}{l(l+1) + \mu^2} \delta_{l'l} \delta_{m'm} \tag{56}$$

where the admissible values of l and m for $\mathcal{A}_\infty$ are $l = 0, 1, 2, \dots, m = 0, 1, \dots, l$, whereas in the case of $\mathcal{A}_N$ they are $l = 0, 1, \dots, N$, $m = 0, 1, \dots, l$.

(b) *Vertex:*

$$V_{l_1 m_1, \dots, l_k m_k} = g_k I_\infty(Y_{l_1 m_1} \cdots Y_{l_k m_k}) \qquad \text{for} \quad \mathcal{A}_\infty \tag{57}$$

$$V_{l_1 m_1, \dots, l_k m_k} = g_k I_N(Y_{l_1 m_1} \cdots Y_{l_k m_k}) \qquad \text{for} \quad \mathcal{A}_N \tag{58}$$

(c) Finally, the summation over all *internal* indices should be performed.

This procedure leads for $\mathcal{A}_\infty$ to finite Feynman diagrams except for diagrams containing the tadpole contribution

$$T_\infty \equiv \sum_{lm} \langle a_{lm} a^*_{lm} \rangle \sim \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{l(l+1) + \mu^2} = \infty$$

This divergence is closely related to the divergence of the propagator

$$G(x, y) = \sum_{lm} \frac{1}{l(l+1) + \mu^2} Y_{lm}(x) Y^*_{lm}(y)$$

in the x representation at points $x = y$. This requires, of course, the regularization of $G(x, y)$, which is, in our case, simply a cutoff in the l-summations.

Indeed, for $\mathcal{A}_N$ all diagrams are obviously finite (since all summations are finite). In particular the tadpole contribution reads

$$T_N = \sum_{l=0}^{N} \sum_{m=-l}^{l} \frac{1}{l(l+1)+\mu^2} \sim \ln N$$

For practical applications an effective method for the calculation of vertex coefficients $V_{l_1m_1,\ldots,l_km_k}$ is needed, both in the standard and noncommutative cases. We shall describe the latter. Since the multiplication by Ψ_{lm} acts in the algebra $\mathcal{A}_N$ as an irreducible tensor operator, we can apply the Wigner–Eckart theorem. Then the product $\Psi_{l_1m_1}(\hat{x})\Psi_{l_2m_2}(\hat{x})$ can be expressed as

$$\Psi_{l_1m_1}(\hat{x})\Psi_{l_2m_2}(\hat{x}) = \sum_{l=|l_1-l_2|}^{l_1+l_2} (l_1m_1,\, l_2m_2|lm)(l_1l_2\|l)\Psi_{lm}(\hat{x}) \tag{59}$$

where $m = -m_1 + m_2$, $(l_1m_1,\, l_2m_2 | lm)$ is a Clebsch–Gordan coefficient, and the symbol $(l_1l_2\|l)$ denotes the so-called reduced matrix element (and depends on the particular algebra in question). Introducing the noncommutative Legendre polynomials $P_l(\xi) = \Psi_{l0}(\hat{x})$, $\xi = \rho^{-1}\hat{x}_3$, we find that the previous equation leads to the coupling rule

$$P_{l_1}(\xi)P_{l_2}(\xi) = \sum_{l=|l_1-l_2|}^{l_1+l_2} (l_10,\, l_20|l0)(l_1l_2\|l)P_l(\xi) \tag{60}$$

The repeated application of (59) then allows us to calculate the required vertices.

Note. The well-known explicit formula for the usual Legendre polynomials allows us to calculate the reduced matrix elements

$$(l_1l_2\|l) = (l_10,\, l_20|l0)$$

that enter the coupling rule in the algebra $\mathcal{A}_\infty$ in terms of a particular Clebsch–Gordan coefficient. Similarly, the explicit formula for the noncommutative Legendre polynomials presented in the Appendix allows us to deduce the reduced matrix elements entering the coupling rule in the algebra $\mathcal{A}_N$.

4. CONCLUDING REMARKS

We have demonstrated that the interacting scalar field on the noncommutative sphere represents a quantum system which has the following properties:

1. The model has a full space symmetry—the full symmetry under isometries (rotations) of the sphere S^2. This is exactly the same symmetry that the interacting scalar field has on the standard sphere.

2. The field has only a finite number of modes. Then the number of degrees of freedom is finite, which leads to the nonperturbative UV regularization, i.e., all quantum mean values of polynomial field functionals are well defined and finite.

Consequently, all Feynman diagrams in the perturbative expansion are finite, even the diagrams containing the tadpole diagram, which are divergent in the model on a standard sphere. Technically, the tadpole is finite because of the cutoff in the number of modes. In our approach, the UV cutoff in the number of modes is supplemented with a highly nontrivial vertex modification [compare equations (57) and (58)]. Moreover, our UV regularization is nonperturbative and is completely determined by the algebra $\mathcal{A}_N$. It originates in the short-distance structure of the space, and does not depend on the field action of the model in question. From the point of view presented above, it would be desirable to analyze a quantization of the models on a noncommutative sphere S^2 containing spinor or gauge fields. In the standard case such models have a more complicated structure of divergences. It is evident that our approach will lead again to a nonperturbative UV regularization.

The usual divergences will appear only in the limit $N \to \infty$. It would be very interesting to isolate the large-N behavior nonperturbatively. By this we mean the Wilson-like approach in which the renormalization group flow in the space of Lagrangians is studied. This can lead to a better understanding of the origin and properties of divergences in quantum field theory. Another interesting direction of research would consist in making connection with matrix models, where, from the technical point of view, very similar integrals have been studied. We strongly believe that qualitatively just the same situation will occur on the four-dimensional sphere S^4, too. Investigations in all these directions are underway.

APPENDIX

With respect to the scalar product

$$(P_l, P_m)_N = I_N(P_l P_m) = \delta_{lm}$$

we define the truncated Legendre polynomials

$$P_l(\xi) = \xi^l a_0^l + \xi^{l-2} a_1^l + \dots, \qquad l = 0, 1, \dots, N$$

as orthonormal. Here the noncommutative integral is given as [see equation (19)]

$$I_N(\xi^n) = \sum_{k=0}^{N} \frac{1}{N+1} \xi_k^n$$

where $\xi_k = [N/(N+2)]^{1/2}(2k/N - 1)$. The polynomials $P_l(\xi)$ can be obtained from the recurrence relation

$$P_{m+1}(\xi) = \frac{1}{a_m} [\xi P_m(\xi) - c_m P_{m-1}(\xi)]$$

where $c_m = I(\xi P_m P_{m-1})$ and $a_m = [I_N(\xi^2 P_m^2) - c_m^2]^{1/2}$.

The $\mathcal{A}_N$-valued truncated spherical functions $\Psi_{lm}(\hat{x})$ satisfy the equation

$$J_i^2 \Psi_{lm}(\hat{x}) = l(l + 1)\Psi_{lm}(\hat{x})$$

Putting $P_l(\xi) = \Psi_{l0}(\hat{x})$, $\xi = \hat{x}_3$, we find that the last equation reduces to a difference equation for the truncated Legendre polynomials

$$(1 - \xi^2) \frac{P_l(\xi + \lambda) - 2P_l(\xi) + P_l(\xi - \lambda)}{\lambda^2} + 2\xi \frac{P_l(\xi + \lambda) - P_l(\xi - \lambda)}{2\lambda} + l(l + 1)P_l(\xi) = 0$$

where $\lambda = 2/[N(N + 2)]^{1/2}$. This equation leads to the recurrence relation for the coefficients a_s^l appearing in the Legendre polynomials:

$$a_s^l = -\frac{1}{s(2l - 2s + 1)} \sum_{r=0}^{s-1} a_r^l \left[\binom{l - 2r}{l - 2s} - \lambda^2 \binom{l - 2r + 1}{l - 2s + 1}\right] \lambda^{2s-2r-2}$$

In the limit $N \to \infty$ (or equivalently $\lambda \to 0$), all formulas reduce to the standard expressions valid for usual Legendre polynomials.

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